Nonlinear Sigma Model for a Condensate Composed of Fermionic Atoms

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Abstract

A nonlinear sigma model is derived for the time development of a Bose-Einstein condensate composed of fermionic atoms. Spontaneous symmetry breaking of a $Sp(2)$ symmetry in a coherent state path integral with anticommuting fields yields Goldstone bosons in a $Sp(2)\backslash U(2)$ coset space. After a Hubbard-Stratonovich transformation from the anticommuting fields to a local self-energy matrix with anomalous terms, the assumed short-ranged attractive interaction reduces this symmetry to a $SO(4)\backslash U(2)$ coset space with only one complex Goldstone field for the singlett pairs of fermions. This bosonic field for the anomalous term of fermions is separated in a gradient expansion from the density terms. The $U(2)$ invariant density terms are considered as a background field or unchanged interacting Fermi sea in the spontaneous symmetry breaking of the $SO(4)$ invariant action and appear as coefficients of correlation functions in the nonlinear sigma model for the Goldstone boson. The time development of the condensate composed of fermionic atoms results in a modified Sine-Gordon equation.

Keywords Bose-Einstein condensation, spontaneous symmetry breaking, coherent states.
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1 Introduction

Experiments of Bose-Einstein condensation (BEC) with bosonic constituents have been realized under various conditions. In many cases the Gross-Pitaevskii (GP) equation with a bosonic field $\psi_{\vec{x}}(t)$ as the order parameter and wavefunction can be applied

$$\imath\hbar\frac{\partial\psi_{\vec{x}}(t)}{\partial t}=-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi_{\vec{x}}(t)+u(\vec{x},t)\;\psi_{\vec{x}}(t)+2\sum_{\vec{x}'}|\psi_{\vec{x}'}(t)|^2\;V_{|\vec{x}'-\vec{x}|}\;\psi_{\vec{x}}(t), \tag{1}$$

where $u(\vec{x},t)$ refers to a time dependent external potential, including the trap potential, and $V_{|\vec{x}'-\vec{x}|}$ is a short-ranged interaction [1, 2]. Transfering the GP equation (1) to the case with fermionic atoms, a coherent field equation with anticommuting numbers can be introduced where the classical field $\psi_{\vec{x}}(t)$ is replaced by a Grassmann-valued field $\chi_{\vec{x},s}(t)$ with spin $s=\uparrow,\downarrow$ [3]

$$\begin{aligned}\imath\hbar\frac{\partial\chi_{\vec{x},s}(t)}{\partial t} &= -\frac{\hbar^2}{2m}\vec{\nabla}^2\chi_{\vec{x},s}(t)+u(\vec{x},t)\;\chi_{\vec{x},s}(t)+ \\ &+ 2\sum_{\vec{x}',s'}\chi^*_{\vec{x}',s'}(t)\;V_{|\vec{x}'-\vec{x}|}\;\chi_{\vec{x}',s'}(t)\;\chi_{\vec{x},s}(t).\end{aligned} \tag{2}$$

As its bosonic counterpart, the Grassmann-valued equation is integrable for a contact interaction and possesses a set of infinite independent integrals of motion [4]. This has been demonstrated by the method of Lax-pairs and r-matrix methods.

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In this paper an effective nonlinear sigma model of bosonic fields for a condensate composed of fermionic atoms is derived for the following quantum Hamiltonian of Fermi operators $\psi_{\vec{x},s}$ corresponding to the classical equation (2) with anticommuting fields $\chi_{\vec{x},s}(t)$ [2]

$$\begin{aligned}\mathcal{H}(\psi^{+},\psi,t) &= \sum_{\vec{x},s;\vec{x}',s'} \psi^{+}_{\vec{x}',s'}\; H_{\vec{x}',s';\vec{x},s}(t)\; \psi_{\vec{x},s} \\ &+ \sum_{\vec{x},s;\vec{x}',s'} \psi^{+}_{\vec{x}',s'}\; \psi^{+}_{\vec{x},s}\; V_{|\vec{x}'-\vec{x}|}\; \psi_{\vec{x},s}\; \psi_{\vec{x}',s'} \\ &+ \sum_{\vec{x}} 2\left(\widetilde{j}^{*}_{\vec{x}}(t)\; \psi_{\vec{x},\uparrow}\; \psi_{\vec{x},\downarrow} + \psi^{+}_{\vec{x},\downarrow}\; \psi^{+}_{\vec{x},\uparrow}\; \widetilde{j}_{\vec{x}}(t)\right)\end{aligned} \tag{3}$$

$$H_{\vec{x}',s';\vec{x},s}(t) = \delta_{s',s}\; \delta_{\vec{x}',\vec{x}}\left(\frac{\vec{p}^{2}}{2m} + v(\vec{x},t)\right) \tag{4}$$

$$v(\vec{x},t) = u(\vec{x},t) - \mu_0 \;\; . \tag{5}$$

The external potential $u(\vec{x},t)$ is shifted by the transformation $\exp\{-\imath/\hbar\cdot\mu_0\; t\}\; \psi_{\vec{x},s}$ with the chemical potential μ_0. This shift is performed because the time derivative is considered as a perturbation in a gradient expansion so that rapid oscillations of the fields do not appear. Since we also include a time dependence in the external potential $u(\vec{x},t)$, the chemical potential μ_0 is in general not an equlibrium value and can be extended with an appropriate adiabatic time dependence $\mu(t)$. However, we assume that the interacting Fermi sea is not strongly perturbed by the time dependence of $u(\vec{x},t)$. The interaction $V_{|\vec{x}'-\vec{x}|} < 0$ is attractive and has to be short-ranged in order to obtain local sigma matrices for the self-energies. Furthermore, the densities of the interacting Fermi sea are regarded as given background fields which are considered as given coefficients for the gradients of the bosonic nonlinear sigma model. The spatial gradient expansion for the nonlinear sigma model is combined with a kind of Thomas-Fermi approximation[2] where the derivative terms of the kinetic energy are taken into account as a perturbation [5]-[7]. In the remainder we investigate and assume a BCS like condensation phenomenon of the atoms, derived from spontaneous symmetry breaking with the source field $\widetilde{j}^{*}_{\vec{x}}(t)$, and exclude the formation of single bosons from bound pairs of atoms because the attractive potential is taken sufficiently short-ranged. In the case of a box potential with depth $V_0 < 0$ and range r_0, this means that $m\;|V_0|\;r_0^2/\hbar^2 < \pi^2/8$ has to be sufficiently small [8]. This case with short-ranged attractive interaction is different from the formation of excitons and biexcitons of the long-ranged Coulomb potential in semiconductors where the Hamiltonian for the strongly bound electrons in semiconductors can be reduced to purely bosonic operators [9]. A review of the nonlinear sigma model in superconducting systems with *delta-function* correlated disordered potentials and the replica-trick can be found in Refs. [10, 11].

A coherent state path integral [12]-[19] with Grassmann fields $\chi_{\vec{x},s}(t_p) = \chi_{\vec{y}}(t_p)$ $(\vec{y} = \{\vec{x},s\})$[3] is used on a nonequilibrium time contour $\int_C \mathrm{d}t_p \ldots = \int_{-\infty}^{\infty}\mathrm{d}t_+ \ldots + \int_{\infty}^{-\infty}\mathrm{d}t_- \ldots$ to express the time development of the system with Hamiltonian (3)

$$\begin{aligned} Z[\mathcal{J}] &= \int \mathrm{d}[\chi_{\vec{y}}(t_p)] \\ &\exp\Big\{-\frac{\imath}{\hbar}\int_C \mathrm{d}t_p \sum_{\vec{y};\vec{y}'} \chi^{*}_{\vec{y}'}(t_p)\,\underbrace{\Big[\delta_{\vec{y}';\vec{y}}\Big(-\imath\hbar\frac{\partial}{\partial t_p} - \imath\varepsilon_p\Big) + H_{\vec{y}';\vec{y}}(t_p)\Big]}_{\widetilde{\mathcal{H}}_{\vec{x}',s';\vec{x},s}(t_p)}\,\chi_{\vec{y}}(t_p)\Big\} \\ \times\;\; &\exp\Big\{-\frac{\imath}{\hbar}\int_C \mathrm{d}t_p \sum_{\vec{y};\vec{y}'} \chi^{*}_{\vec{y}'}(t_p)\;\chi^{*}_{\vec{y}}(t_p)\;V_{|\vec{x}'-\vec{x}|}\;\chi_{\vec{y}}(t_p)\;\chi_{\vec{y}'}(t_p)\Big\}\end{aligned} \tag{6}$$

[2]The spatial sum $\sum_{\vec{x}}\ldots$ is dimensionless and is scaled with the system volume so that $\sum_{\vec{x}}\ldots$ is equivalent to $\int_{L^d}\mathrm{d}^d x/L^d\ldots$.

[3]In the following $\vec{y}$-vectors refer to the combined spatial vector $\vec{x}$ and spin variable $s=\uparrow,\downarrow$ as abbreviation.

$$
\begin{aligned}
& \times \quad \exp\Big\{ -\frac{\imath}{\hbar}\int_C \mathrm{d}t_p \sum_{\vec{x}} \widetilde{j}^{*}_{\vec{x}}(t_p)\ \Big(\chi_{\vec{x},\uparrow}(t_p)\ \chi_{\vec{x},\downarrow}(t_p) - \chi_{\vec{x},\downarrow}(t_p)\ \chi_{\vec{x},\uparrow}(t_p)\Big) + \text{h.c.}\Big\} \\
& \times \quad \exp\Big\{ -\frac{\imath}{2\hbar}\int_C \mathrm{d}t^{(1)}_{p_1}\ \mathrm{d}t^{(2)}_{p_2} \sum_{\vec{y};\vec{y}'} \eta^{*}_{\vec{y}'}(t^{(2)}_{p_2})\ \mathcal{J}_{\vec{y}';\vec{y}}(t^{(2)}_{p_2};t^{(1)}_{p_1})\ \eta_{\vec{y}}(t^{(1)}_{p_1})\Big\}
\end{aligned}
$$

$$
\chi_{\vec{y}}(t_p) = \chi_{\vec{x},s}(t_p) \tag{7}
$$

$$
\eta_{\vec{y}}(t_p) = \eta_{\vec{x},s}(t_p) = \begin{pmatrix} \chi_{\vec{x},s}(t_p) \\ \chi^{*}_{\vec{x},s}(t_p) \end{pmatrix}. \tag{8}
$$

A source field $\widetilde{j}^{*}_{\vec{x}}(t_p)$ is applied to generate bosonic pairs $\chi_{\vec{x},\uparrow}(t_p)\ \chi_{\vec{x},\downarrow}(t_p)$ out of the vacuum state for spontaneous symmetry breaking of the $U(1)$ invariant one particle part $\widetilde{\mathcal{H}}$ and the $U(1)$ invariant interaction [20, 21]. In order to generate observables, the source term $\mathcal{J}_{\vec{y}';\vec{y}}(t^{(2)}_{p_2};t^{(1)}_{p_1})$ has to be introduced where the fields $\chi_{\vec{x},\uparrow}(t_p)$ and $\chi_{\vec{x},\downarrow}(t_p)$ have to be combined to the four component vector $\eta_{\vec{x},s}(t_p)$ (8) so that pair condensate terms $\chi_{\vec{x}',s'}(t^{(2)}_{p_2})\ \chi_{\vec{x},s}(t^{(1)}_{p_1})$ can be obtained by simple differentiation with respect to the matrix $\mathcal{J}_{\vec{y}';\vec{y}}(t^{(2)}_{p_2};t^{(1)}_{p_1})$ [22]. A nonhermitian infinitesimal part $-\imath\ \varepsilon_p = -\imath\ (\pm\varepsilon)$, $(p=\pm)$ on the time contour has to be included for the analytic and convergence properties of Green functions, derived from the coherent state path integral $Z[\mathcal{J}]$ (6).

Apart from the commutation relations the field equation, Hamiltonian and coherent state path integral are formally similar in terms of Grassmann numbers to the bosonic case [16]. Therefore, variations of the action in (6) and other approximations can be performed, however, compared to the condensation of single bosonic constituents, only a small fraction of fermions can condense near the Fermi energy [1]. This means in terms of a density matrix formulation that expectation values of densities $\langle\chi_{\vec{x}',s'}(t_p)\ \chi^{*}_{\vec{x},s}(t_p)\rangle$ are considerably larger than the pair condensate $\langle\chi_{\vec{x}',s'}(t_p)\ \chi_{\vec{x},s}(t_p)\rangle$. It is the aim of this paper to extract the various densities and pair condensate functions from the coherent state path integral and to derive effective equations for the pair condensate composed of fermionic atoms, in analogy to the GP-equation for bosons (1) or fermions (2). The nonlocal order parameter $\Phi_{\vec{y}';\vec{y}}(t_p)$ has a matrix form

$$
\begin{aligned}
\Phi_{\vec{y}';\vec{y}}(t_p) &= \begin{pmatrix} \chi_{\vec{x}',s'}(t_p) \\ \chi^{*}_{\vec{x}',s'}(t_p) \end{pmatrix} \otimes \begin{pmatrix} \chi^{*}_{\vec{x},s}(t_p), & \chi_{\vec{x},s}(t_p) \end{pmatrix} \qquad (9) \\
&= \begin{pmatrix} \langle\chi_{\vec{x}',s'}(t_p)\ \chi^{*}_{\vec{x},s}(t_p)\rangle & \langle\chi_{\vec{x}',s'}(t_p)\ \chi_{\vec{x},s}(t_p)\rangle \\ \langle\chi^{*}_{\vec{x}',s'}(t_p)\ \chi^{*}_{\vec{x},s}(t_p)\rangle & -\langle\chi_{\vec{x},s}(t_p)\ \chi^{*}_{\vec{x}',s'}(t_p)\rangle \end{pmatrix},
\end{aligned}
$$

where a doubling of the spin space has to be considered because of the source field $\widetilde{j}^{*}_{\vec{x}}(t_p)$ which causes the spontaneous symmetry breaking. The order parameter (9) is invariant under $U(2)$ transformations in spin space which does not alter the block structure into densities and pair condesates. The form of the order parameter also allows a global hyperbolic symmetry which combines densities and pair condensates. A complete symmetry group of the path integral is spontaneously broken by the subgroup $U(2)$ for the invariance of the densities and the source term. This symmetry breaking leads to a nonlinear sigma model after a gradient expansion for the anomalous terms. The various steps for obtaining the nonlinear sigma model are briefly listed :

- coherent state path integral
- transformation of the quartic interaction to densities, anomalous terms and the order parameter
- Hubbard-Stratonovich transformation from the fields to the self-energy and integration over the remaining bilinear anticommuting fields [23, 16]
- the short-ranged attractive interaction reduces the $Sp(2)$ symmetry of the path integral with a *spatially nonlocal* self-energy to a *spatially local* self-energy matrix with $SO(4)$ symmetry

- separation of the self-energy into densities and non-diagonal terms on a coset space according to spontaneous breaking of the orthogonal symmetry $SO(4)\backslash U(2)$ and determination of the measure
- separation of the coherent state path integral into block diagonal $U(2)$ invariant density matrices and anomalous terms including a gradient expansion

2 Hubbard-Stratonovich transformation and self-energy

The quartic interaction with the short-ranged two body potential has to be transformed to a relation with an order parameter similar to $\Phi_{\vec{y}';\vec{y}}(t_p)$ (9). The antisymmetric fields $\chi_{\vec{x},s}(t_p)$, $\chi^*_{\vec{x},s}(t_p)$ and $\chi_{\vec{x}',s'}(t_p)$, $\chi^*_{\vec{x}',s'}(t_p)$ can be combined in the following even matrices $r_{\vec{y}';\vec{y}}(t_p) = r_{\vec{x}',s';\vec{x},s}(t_p)$ and $\rho_{\vec{y}';\vec{y}}(t_p) = \rho_{\vec{x}',s';\vec{x},s}(t_p)$ where r and ρ are hermitian and antisymmetric, respectively

$$\begin{aligned} r_{\vec{y}';\vec{y}}(t_p) &= \chi_{\vec{y}'}(t_p)\,\chi^*_{\vec{y}}(t_p) && (10)\\ r^*_{\vec{y}';\vec{y}}(t_p) &= \chi_{\vec{y}}(t_p)\,\chi^*_{\vec{y}'}(t_p) = r_{\vec{y};\vec{y}'}(t_p) \to r^+(t_p) = r(t_p) && (11)\\ \rho_{\vec{y}';\vec{y}}(t_p) &= \chi_{\vec{y}'}(t_p)\,\chi_{\vec{y}}(t_p) = -\chi_{\vec{y}}(t_p)\,\chi_{\vec{y}'}(t_p) = -\rho_{\vec{y};\vec{y}'}(t_p) \to \rho(t_p) = -\rho^T(t_p) && (12)\\ \rho^*_{\vec{y}';\vec{y}}(t_p) &= \chi^*_{\vec{y}}(t_p)\,\chi^*_{\vec{y}'}(t_p) = \rho^{T*}_{\vec{y};\vec{y}'}(t_p) = \rho^+_{\vec{y};\vec{y}'}(t_p)\,. && (13) \end{aligned}$$

In terms of the even matrices r and ρ, the quartic interaction can be written in the form of an order parameter $R^{ab}_{\vec{x},s;\vec{x}',s'}(t_p)$ as in (9) with a doubling of the dimension of spin space where the superscripts $a,\ b = 1,2$ refer to the doubling and $s,\ s' = \uparrow,\downarrow$ to the spins

$$\begin{aligned} &\sum_{\vec{x},s;\vec{x}',s'} \chi^*_{\vec{x}',s'}(t_p)\,\chi^*_{\vec{x},s}(t_p)\,\chi_{\vec{x},s}(t_p)\,\chi_{\vec{x}',s'}(t_p)\,V_{|\vec{x}'-\vec{x}|} = && (14)\\ &= \frac{1}{4}\sum_{\vec{y};\vec{y}'}\Big(\chi^*_{\vec{y}'}(t_p)\,\chi_{\vec{y}'}(t_p) - \chi_{\vec{y}'}(t_p)\,\chi^*_{\vec{y}'}(t_p)\Big)\Big(\chi^*_{\vec{y}}(t_p)\,\chi_{\vec{y}}(t_p) - \chi_{\vec{y}}(t_p)\,\chi^*_{\vec{y}}(t_p)\Big)\,V_{|\vec{x}'-\vec{x}|}\\ &= -\frac{1}{4}\sum_{\vec{x},\vec{x}'} V_{|\vec{x}'-\vec{x}|}\underset{s,s',a,b}{\mathrm{Tr}}\left[R^{ab}_{\vec{x},s;\vec{x}',s'}(t_p)\begin{pmatrix}1_2 & \\ & -1_2\end{pmatrix}R^{ba}_{\vec{x}',s';\vec{x},s}(t_p)\begin{pmatrix}1_2 & \\ & -1_2\end{pmatrix}\right] \end{aligned}$$

$$\begin{aligned} R^{ab}_{\vec{x},s;\vec{x}',s'}(t_p) &= \begin{pmatrix}\chi_{\vec{x},s}(t_p)\\ \chi^*_{\vec{x},s}(t_p)\end{pmatrix}\otimes\begin{pmatrix}\chi^*_{\vec{x}',s'}(t_p), & \chi_{\vec{x}',s'}(t_p)\end{pmatrix} && (15)\\ &= \begin{pmatrix} r_{\vec{x},s;\vec{x}',s'}(t_p) & \rho_{\vec{x},s;\vec{x}',s'}(t_p)\\ \rho^+_{\vec{x},s;\vec{x}',s'}(t_p) & -r^T_{\vec{x},s;\vec{x}',s'}(t_p)\end{pmatrix}. \end{aligned}$$

Obviously, the quartic interaction can be expressed with the nonlocal order parameter $R^{ab}_{\vec{x},s;\vec{x}',s'}(t_p)$ so that a global hyperbolic symmetry results with the diagonal matrix $\kappa = \mathrm{diag}(1,1,-1,-1)$ and the matrix T between densities and pair condensates ($T^+\,\kappa\,T = \kappa$), apart from a $U(2)$ invariance in spin space

$$\begin{aligned} R &\to T\,R\,T^+ && (16)\\ T &= \begin{pmatrix}\sqrt{1+t^+t} & t^+\\ t & \sqrt{1+tt^+}\end{pmatrix} \qquad t := t_{ss'} \qquad t = t^T && (17) \end{aligned}$$

$$T^+\underbrace{\begin{pmatrix}1_2 & \\ & -1_2\end{pmatrix}}_{\kappa}T = \underbrace{\begin{pmatrix}1_2 & \\ & -1_2\end{pmatrix}}_{\kappa}\ . \qquad (18)$$

The 2×2 matrix $t_{ss'}$ in spin space of the 4×4 matrix $T^{ab}_{ss'}$ has to be complex symmetric and therefore contains 6 real parameters (see appendix A). The hyperbolic symmetry with matrix

T and the $U(2)$ invariance in spin space is equivalent to a symplectic symmetry group $Sp(N/2)$ and not a unitary group as $U(N/2,N/2)$ because of the number of independent parameters which equals ten in the considered case. The symplectic symmetry becomes obvious after an exchange of the first, second with the third, fourth row of the diagonal matrix κ and reordering the matrix T^+ to its transpose T^T in relation (18) (see appendix A). Since the matrix T has dimensions 4×4 and the number of independent parameters for $Sp(N/2)$ is $\frac{1}{2}N(N+1)$, a $Sp(2)$ invariance is obtained for relation (14). Using the identity for the Hubbard-Stratonovich transformation with the self-energy matrix $\Sigma^{ab}_{\vec{x},s;\vec{x}',s'}(t_p)$ consisting of commuting elements only

$$\Sigma^{ab}_{\vec{x},s;\vec{x}',s'}(t_p)=\left(\begin{array}{cc} s_{\vec{x},s;\vec{x}',s'}(t_p) & \sigma_{\vec{x},s;\vec{x}',s'}(t_p) \\ \sigma^{+}_{\vec{x},s;\vec{x}',s'}(t_p) & -s^{T}_{\vec{x},s;\vec{x}',s'}(t_p)\end{array}\right) \qquad s=s^{+}\quad \sigma=-\sigma^{T}\ , \tag{19}$$

the quartic interaction term with $V_{|\vec{x}'-\vec{x}|}$ can be expressed as a quadratic term of the self-energy and a bilinear product of anticommuting fields $\eta_{\vec{x},s}(t_p)=(\chi_{\vec{x},s}(t_p),\chi^{*}_{\vec{x},s}(t_p))^{T}$

$$\begin{aligned}
&\exp\Big\{-\frac{\imath}{\hbar}\int_C dt_p\sum_{\vec{y};\vec{y}'}\chi^{*}_{\vec{y}'}(t_p)\,\chi^{*}_{\vec{y}}(t_p)\,V_{|\vec{x}'-\vec{x}|}\,\chi_{\vec{y}}(t_p)\,\chi_{\vec{y}'}(t_p)\Big\}=\int d[s]\,d[\sigma] \\
&\quad\exp\Big\{-\frac{\imath}{4\hbar}\int dt_p\sum_{\vec{x};\vec{x}'}\frac{1}{V_{|\vec{x}'-\vec{x}|}}\mathop{\mathrm{Tr}}_{s,s',a,b}\Big[\Sigma^{ab}_{\vec{x},s;\vec{x}',s'}(t_p)\,\kappa^{bb}\,\Sigma^{ba}_{\vec{x}',s';\vec{x},s}(t_p)\,\kappa^{aa}\Big]\Big\} \\
&\quad\exp\left\{-\frac{\imath}{2\hbar}\int_C dt_p\sum_{\vec{y};\vec{y}'}\left(\begin{array}{c}\chi_{\vec{y}}(t_p)\\ \chi^{*}_{\vec{y}}(t_p)\end{array}\right)^{+}\ \Sigma_{\vec{y};\vec{y}'}(t_p)\ \left(\begin{array}{c}\chi_{\vec{y}'}(t_p)\\ \chi^{*}_{\vec{y}'}(t_p)\end{array}\right)\right\} \\
&\quad\kappa=\mathrm{diag}(1,1,-1,-1)\ .
\end{aligned}\tag{20}$$

Substitution of the interaction with the Hubbard-Stratonovich transformation yields the following coherent state path integral where a doubling of the one particle terms has to be performed because of the source terms so that a bilinear product of anticommuting fields is obtained

$$\begin{aligned}
Z[\mathcal{J}]=&\int d[s]\,d[\sigma]\ \exp\left\{-\frac{\imath}{4\hbar}\int_C dt_p\sum_{\vec{x},\vec{x}'}\frac{1}{V_{|\vec{x}'-\vec{x}|}}\mathop{\mathrm{Tr}}_{s,s',a,b}\Big[\Sigma\,\kappa\,\Sigma\,\kappa\Big]\right\} \\
\times&\int d[\chi]\ \exp\left\{-\frac{\imath}{2\hbar}\int_C dt_p\,\eta^{+}\left[\left(\begin{array}{cc}\widetilde{\mathcal{H}} & -j\\ -j^{+} & -\widetilde{\mathcal{H}}^{T}\end{array}\right)+\mathcal{J}+\Sigma\right]\,\eta\right\}
\end{aligned}\tag{21}$$

$$j_{ss'}(\vec{x},t_p)\;=\;\left(\begin{array}{cc}0 & \widetilde{j}(\vec{x},t_p)\\ -\widetilde{j}(\vec{x},t_p) & 0\end{array}\right) \tag{22}$$

$$\widetilde{\mathcal{H}}_{ss'}(t_p)\;=\;\delta_{ss'}\left(-\imath\hbar\frac{\partial}{\partial t_p}-\imath\,\varepsilon_p+\frac{\vec{p}^{2}}{2m}+v(\vec{x},t)\right). \tag{23}$$

After integration over the anticommuting variables, the path integral only contains the self-energy Σ with the hermitian matrix s for the density terms and the antisymmetric matrix σ for the anomalous terms

$$Z[\mathcal{J}]\;=\;\frac{\int d[s]\,d[\sigma]\ \exp\left\{-\frac{\imath}{4\hbar}\int_C dt_p\sum_{\vec{x},\vec{x}'}\frac{1}{V_{|\vec{x}'-\vec{x}|}}\mathop{\mathrm{Tr}}_{s,s',a,b}\Big[\Sigma\,\kappa\,\Sigma\,\kappa\Big]\right\}}{\sqrt{\det\left[\left(\begin{array}{cc}\widetilde{\mathcal{H}}(t_p) & -j(t_p)\\ -j^{+}(t_p) & -\widetilde{\mathcal{H}}^{T}(t_p)\end{array}\right)+\mathcal{J}+\left(\begin{array}{cc}s(t_p) & -\sigma(t_p)\\ -\sigma^{+}(t_p) & -s^{T}(t_p)\end{array}\right)\right]}}. \tag{24}$$

The generating function $Z[\mathcal{J}]$ (24) is invariant under a symplectic symmetry group $Sp(2)$ which has its cause in the anticommuting properties of the fields $\chi_{\vec{x},s}(t_p)$ and the doubling of spin space (see appendix A). This symplectic invariance is spontaneously broken by the $U(2)$ invariance of the matrix s and $-s^T$ and the source term $j(t_p)$ (22) so that three complex or six real Goldstone fields result on the coset space $Sp(2)\backslash U(2)$ because the corresponding dimensions of the Lie algebras for $Sp(2)$ and $U(2)$ are ten and four, respectively. The densities s and $-s^T$ represent the self-energy of the interacting Fermi sea and can be regarded as *a vacuum state or background field* on which the subgroup $U(2)$ invariantly acts so that the symmetry $Sp(2)$ of the complete Lagrangian is spontaneously broken to three complex Goldstone fields.

However, if the trap potential in $v(\vec{x},t)$ can be restricted to a typical distance a_0 and if this distance a_0 is considerably larger than the range $r_0 \approx |\vec{x}'-\vec{x}|$ of the two body potential $V_{|\vec{x}'-\vec{x}|}$, there are strong oscillations in the quadratic term with the nonlocal self-energy $\Sigma^{ab}_{\vec{x},s;\vec{x}',s'}(t_p)$ of (24) because $1/V_{|\vec{x}'-\vec{x}|}$ tends to infinity as the short ranged potential $V_{|\vec{x}'-\vec{x}|}$ approaches zero. Therefore, the spatially local parts $\Sigma^{ab}_{ss'}(\vec{x},t_p)$ of the self-energy are only retained in the path integral (24)

$$\frac{1}{V_{|\vec{x}'-\vec{x}|}} \quad\to\quad \infty \qquad \text{for} \quad |\vec{x}'-\vec{x}| > r_0 \tag{25}$$

$$\Sigma^{ab}_{\vec{x},s;\vec{x}',s'}(t_p) \quad\to\quad \Sigma^{ab}_{ss'}(\vec{x},t_p)\;\;\delta_{\vec{x}',\vec{x}}\;\;. \tag{26}$$

This can be accomplished by integration over the matrix elements $\Sigma^{ab}_{\vec{x},s;\vec{x}',s'}(t_p)$ with $|\vec{x}-\vec{x}'| > r_0$ in (24) and eliminates the nonlocal parts in the self-energy which cause the strong oscillation on the nonequilibrium time contour in (24). The diagonal parts $\sigma_{\vec{x},\uparrow;\vec{x}',\uparrow}(t_p)$, $\sigma_{\vec{x},\downarrow;\vec{x}',\downarrow}(t_p)$ of spin space in the matrix $\sigma(t_p)$ tend to zero because of the antisymmetry of the pair condensate in the spatial part. The vanishing of the diagonal elements of spin space in $\sigma(t_p)$, due to the assumed short-ranged and spin independent interaction $V_{|\vec{x}-\vec{x}'|}$, corresponds to the observation that there is usually no triplett pairing of fermions in the condensate. Consequently, only one complex Goldstone field for the singlett mode remains whereas the other two complex fields, which result from spontaneous symmetry breaking $Sp(2)\backslash U(2)$ in the path integral with nonlocal self-energy, are suppressed because of the short-ranged interaction.[4] A $Sp(2)\backslash U(2)$ coset space for the Goldstone bosons would result if the interaction $V_{|\vec{x}'-\vec{x}|}$ was constant for any pair of spatial vectors $\vec{x}'$, $\vec{x}$ so that every atom would interact with equal weight with all other atoms independent of distance.

After a shift of the self-energy matrix by the spontaneous symmetry breaking term, the coherent state path integral $Z[\mathcal{J}]$ (24) is transformed to a local self-energy $\Sigma^{ab}_{ss'}(\vec{x},t_p)$ consisting of the hermitian local matrix s and the antisymmetric local matrix σ in spin space and the two body potential restricted to a finite typical zero distance value $V_0 < 0$ [5]

$$\begin{aligned} Z[\mathcal{J}] = \int & d[s]\; d[\sigma] \qquad\qquad (27) \\ & \exp\left\{\frac{1}{2}\int_C dt_p \sum_{\vec{x}} \mathcal{N} \underset{s,s',a,b}{\mathrm{Tr}} \ln\left[\begin{pmatrix} \widetilde{\mathcal{H}} & 0 \\ 0 & -\widetilde{\mathcal{H}}^T \end{pmatrix} + \mathcal{J} + \begin{pmatrix} s(t_p) & -\sigma(t_p) \\ -\sigma^+(t_p) & -s^T(t_p) \end{pmatrix}\right]\right\} \times \\ \times\quad & \exp\left\{-\frac{\imath}{4\hbar}\int_C dt_p \sum_{\vec{x}} \frac{1}{V_0} \underset{s,s',a,b}{\mathrm{Tr}} \left[\left(\Sigma + \begin{pmatrix} 0 & j \\ j^+ & 0 \end{pmatrix}\right)\; \kappa\; \left(\Sigma + \begin{pmatrix} 0 & j \\ j^+ & 0 \end{pmatrix}\right)\; \kappa\right]\right\}. \end{aligned}$$

The 4×4 local self-energy matrix in (27) consists of the hermitian $U(2)$ invariant density term $s(t_p)$ with four real parameters and the antisymmetric 2×2 anomalous term with one complex field so that the matrix $\Sigma^{ab}_{ss'}(\vec{x},t_p)$ contains six real fields. The number of independent parameters

[4]In the case of spin dependent forces or other interactions $V_{|\vec{x}-\vec{x}'|}$ which have their maximum for $|\vec{x}-\vec{x}'| \neq 0$, but vanishing zero distance interaction V_0, other complex Goldstone fields have to be chosen. These cases are excluded in the present paper.

[5]The variable $\mathcal{N}$ in (27) is a normalization factor $\mathcal{N} = (L/\Delta x)^d \cdot (1/\Delta t)$ because a determinant without integration measure is considered in the Tr ln term.

and the dimension of $\Sigma_{ss'}^{ab}(\vec{x},t_p)$ indicate a $SO(4)$ symmetry where the antisymmetry of $SO(4)$ generators becomes obvious after exchange of the first, second with the third, fourth row of the 4×4 matrices in (27). The following *local* parametrization (31-33) of the self-energy with only one complex field $\phi(\vec{x},t_p)$ as Goldstone boson can be chosen in the coset space $SO(4)\backslash U(2)$ in order to separate the anomalous term from the unchanged interacting Fermi sea with density matrix s, $-s^T$ in the spontaneous symmetry breaking

$$\begin{aligned}
\Sigma_{ss'}^{ab}(\vec{x},t_p) &= \begin{pmatrix} s_{ss'}(\vec{x},t_p) & \sigma_{ss'}(\vec{x},t_p) \\ \sigma_{ss'}^{+}(\vec{x},t_p) & -s_{ss'}^{T}(\vec{x},t_p) \end{pmatrix} && (28)\\
s_{s's}^{*}(\vec{x},t_p) &= s_{ss'}(\vec{x},t_p) \qquad \sigma_{s's}(\vec{x},t_p) = -\sigma_{ss'}(\vec{x},t_p) && (29)\\
\sigma_{\uparrow\uparrow}(\vec{x},t_p) &= \sigma_{\downarrow\downarrow}(\vec{x},t_p) = 0 && (30)
\end{aligned}$$

$$\begin{aligned}
\Sigma_{ss'}^{ab}(\vec{x},t_p) &= T(\vec{x},t_p) \begin{pmatrix} s_D(\vec{x},t_p) & 0 \\ 0 & -s_D^T(\vec{x},t_p) \end{pmatrix} T^{+}(\vec{x},t_p) && (31)\\
T &= \begin{pmatrix} \sqrt{1+t^{+}t} & t^{+} \\ t & \sqrt{1+tt^{+}} \end{pmatrix} \qquad s_D = \begin{pmatrix} u & w \\ w^{*} & v \end{pmatrix} && (32)\\
t &= \phi(\vec{x},t_p)\;1_2 \qquad w = w_r + \imath\, w_i \qquad u,v,w_r,w_i \in \mathsf{R}\ . && (33)
\end{aligned}$$

The coherent state path integral can be transformed with the chosen parametrization (31-33) of Σ and the change of integration measure $w_i^2(\vec{x},t_p)/4$ to the form

$$\begin{aligned}
Z[\mathcal{J}] = \int \mathrm{d}[u]\;\mathrm{d}[v]\;\mathrm{d}[w_r]\;\mathrm{d}[w_i]\;\mathrm{d}[\phi]\;\Bigg(\prod_{\{\vec{x},t_p\}} \frac{w_i^2(\vec{x},t_p)}{4}\Bigg) \qquad (34)\\
\exp\Bigg\{-\frac{\imath}{4\hbar}\int_C \mathrm{d}t_p \sum_{\vec{x}} \frac{1}{V_0}\Bigg(2\;\underset{s,s'}{\mathrm{tr}}\,(s_D^2) - 4\,\widetilde{j}\,\widetilde{j}^{*} - 16\,w_i\,\Im(\phi\,\widetilde{j})\,\sqrt{1+|\phi|^2}\Bigg)\Bigg\}\\
\exp\Bigg\{\frac{1}{2}\int_C \mathrm{d}t_p \sum_{\vec{x}} \mathcal{N}\underset{s,s',a,b}{\mathrm{Tr}}\;\ln\Bigg[\begin{pmatrix} \widetilde{\mathcal{H}} & \\ & -\widetilde{\mathcal{H}}^{T} \end{pmatrix} + \mathcal{J} + T\begin{pmatrix} s_D & \\ & -s_D^T \end{pmatrix} T^{+}\Bigg]\Bigg\}\ .
\end{aligned}$$

Since only a small fraction of the Fermi sea condenses, classical equations for the 2×2 density matrix $s_D(\vec{x},t_p)$ can be obtained by variation of the action in (34) with respect to u, v, w_r and w_i (32,33) where the matrix T is set to the unit matrix and the integration measure $w_i^2(\vec{x},t_p)$ is included in the variation[6]

$$\begin{aligned}
&\text{variation}\;\; \delta u(\vec{x},t_p)\ : && (35)\\
&\quad -\frac{\imath}{V_0}\,u^0 + (\widetilde{\mathcal{H}}+s_D^0)^{-1}_{\uparrow\uparrow}(\vec{x},t_p) = 0\\
&\text{variation}\;\; \delta v(\vec{x},t_p)\ : && (36)\\
&\quad -\frac{\imath}{V_0}\,v^0 + (\widetilde{\mathcal{H}}+s_D^0)^{-1}_{\downarrow\downarrow}(\vec{x},t_p) = 0\\
&\text{variation}\;\; \delta w_r(\vec{x},t_p)\ : && (37)\\
&\quad -\frac{2\imath}{V_0}\,w_r^0 + (\widetilde{\mathcal{H}}+s_D^0)^{-1}_{\uparrow\downarrow}(\vec{x},t_p) + (\widetilde{\mathcal{H}}+s_D^0)^{-1}_{\downarrow\uparrow}(\vec{x},t_p) = 0\\
&\text{variation}\;\; \delta w_i(\vec{x},t_p)\ : && (38)\\
&\quad -\frac{2\imath}{V_0}\,w_i^0 + \imath\Big[(\widetilde{\mathcal{H}}+s_D^0)^{-1}_{\uparrow\downarrow}(\vec{x},t_p) - (\widetilde{\mathcal{H}}+s_D^0)^{-1}_{\downarrow\uparrow}(\vec{x},t_p)\Big] + \frac{2}{w_i^0} = 0\ .
\end{aligned}$$

The classical equations (35-38) for the resulting matrix $s_D^0(\vec{x},t_p)$ can be simplified with the Thomas-Fermi approximation where the kinetic energy can be neglected because of the large atom masses

[6] The fields $u(\vec{x},t_p)$, $v(\vec{x},t_p)$, $w_r(\vec{x},t_p)$, $w_i(\vec{x},t_p)$ in $s_D(\vec{x},t_p)$ have to be scaled by the factor $(\Delta t/\hbar)\cdot(\Delta x/L)^d$ to dimensionless quantities for the variation.

and an assumed homogeneous self-energy of the bulk Fermi sea. This gives algebraic equations for $s_D^0(\vec{x},t_p)$ which can be applied in the correlation functions of the nonlinear sigma model with matrix T (32) following in section 3. Using the parametrization into block diagonal densities s_D for the Fermi sea and anomalous terms, the determinant in (34) has to be expanded with respect to the gradients contained in $\widetilde{\mathcal{H}}$ [5]-[7].

3 Separation into densities and anomalous terms with a gradient expansion

Applying the chosen parametrization (31-33) for Σ, we can insert the term $\kappa\, T\, \kappa\, T^{+}\, \kappa$ into the $\mathrm{Tr}\ln$ term of (34) without a modification of the coherent state path integral because the determinant of κ in combined spin and hyperbolic space equals unity

$$\begin{aligned}\mathcal{O}_1 &= \int_C \mathrm{d}t_p \sum_{\vec{x}} \mathcal{N} \underset{s,s',a,b}{\mathrm{Tr}} \ln\left\{\left[\begin{pmatrix}\widetilde{\mathcal{H}} & \\ & -\widetilde{\mathcal{H}}^T\end{pmatrix} + \mathcal{J} + \Sigma\right] \underbrace{\kappa\, T\, \kappa}_{T^{-1}}\, T^{+}\, \kappa\right\} \qquad (39)\\ &= \int_C \mathrm{d}t_p \sum_{\vec{x}} \mathcal{N} \underset{s,s',a,b}{\mathrm{Tr}} \ln\left[\begin{pmatrix}\widetilde{\mathcal{H}} + s_D & \\ & \widetilde{\mathcal{H}}^T + s_D^T\end{pmatrix} + T\, \kappa\, \mathcal{J}\, T^{-1} + \right.\\ &\left. + \underbrace{T\begin{pmatrix}\widetilde{\mathcal{H}} & \\ & \widetilde{\mathcal{H}}^T\end{pmatrix} T^{-1} - \begin{pmatrix}\widetilde{\mathcal{H}} & \\ & \widetilde{\mathcal{H}}^T\end{pmatrix}}_{\delta\mathcal{H}_{ss'}^{ab}}\right].\end{aligned}$$

The gradient expansion of $\delta\mathcal{H}_{ss'}^{ab}$ gives the following operator $\delta\mathcal{H}_{ss'}^{ab} = \delta_{ss'}\, \delta\mathcal{H}^{ab}$

$$\begin{aligned}& T\begin{pmatrix}\widetilde{\mathcal{H}} & \\ & \widetilde{\mathcal{H}}^T\end{pmatrix} T^{-1} - \begin{pmatrix}\widetilde{\mathcal{H}} & \\ & \widetilde{\mathcal{H}}^T\end{pmatrix} = \qquad (40)\\ &= (T\, \kappa\, T^{-1} - \kappa)\, (-\hat{E}_p) + T\, \kappa\, (-E_p T^{-1})\\ &- \frac{\hbar^2}{2m} (\partial_\mu T)\, T^{-1}\, (\partial_\mu T)\, T^{-1} + \frac{\hbar^2}{2m} (\partial_\mu T)\, T^{-1}\, \hat{\partial}_\mu + \frac{\hbar^2}{2m} \hat{\partial}_\mu\, (\partial_\mu T)\, T^{-1}\\ &\hat{E}_p = \imath\hbar\, \frac{\hat{\partial}}{\partial t_p}, \qquad (41)\end{aligned}$$

where one has to distinguish between the operators $\hat{\partial}_\mu$, $\hat{E}_p$ and the derivatives $(\partial_\mu T)$,$(E_p T) = \imath\hbar(\partial T/\partial t_p)$ of the matrices T, T^{-1}, e.g. $T\, \hat{\partial}_\mu\, T^{-1} = T\left[(\partial_\mu T^{-1}) + T^{-1}\, \hat{\partial}_\mu\right]$.

Expanding the $\mathrm{Tr}\ln$ term $\mathcal{O}_1$ up to second order in $\delta\mathcal{H}_{ss'}^{ab}$, we obtain the expression (42) with the Green function $G_{ss'}^{a}$ of the block diagonal density matrix s_D in spin space in symbolic form (spatial and time coordinates are omitted for brevity). In the following the field w_i in the matrix s_D (32,33) for the Green function $G_{ss'}^{a}$ has to be separated from the expansion because it couples to the orignal $U(1)$ symmetry violating source term $\widetilde{j}$ (34,6)

$$\begin{aligned}\mathcal{O}_1 &= 2\int_C \mathrm{d}t_p \sum_{\vec{x}} \mathcal{N} \underset{s,s'}{\mathrm{tr}} \ln\left(\widetilde{\mathcal{H}} + s_D\right) \qquad (42)\\ &+ \int_C \mathrm{d}t_p \sum_{\vec{x}} \mathcal{N} \underset{s,s',a,b}{\mathrm{Tr}} \left[G_{ss'}^{a}\, (T\, \kappa\, \mathcal{J}\, T^{-1} + \delta\mathcal{H})_{s's}^{aa}\right]\\ &- \frac{1}{2}\int_C \mathrm{d}t_p \sum_{\vec{x}} \mathcal{N} \underset{s,s',a,b}{\mathrm{Tr}} \left[G_{ss'}^{a}\, \delta\mathcal{H}^{ab}\, G_{s's}^{b}\, \delta\mathcal{H}^{ba}\right]\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\int_C \mathrm{d}t_p \sum_{\vec{x}} \mathcal{N} \operatorname*{Tr}_{s,s',a,b}\Big[G^{a_1}_{s_1 s_1'}\;(T\;\kappa\;\mathcal{J}\;T^{-1})^{a_1 a_2}_{s_1' s_2'}\;G^{a_2}_{s_2' s_2}\;(T\;\kappa\;\mathcal{J}\;T^{-1})^{a_2 a_1}_{s_2 s_1}\Big] \\
&- \int_C \mathrm{d}t_p \sum_{\vec{x}} \mathcal{N} \operatorname*{Tr}_{s,s',a,b}\Big[G^{a_1}_{s_1 s_1'}\;(T\;\kappa\;\mathcal{J}\;T^{-1})^{a_1 a_2}_{s_1' s_2'}\;G^{a_2}_{s_2' s_1}\;\delta\mathcal{H}^{a_2 a_1}\Big].
\end{aligned}
$$

The hermitian self-energy $s_{D;ss'}(\vec{x},t_p)$ without the field w_i is only contained in the Green function $G^a_{ss'}(\vec{x},t_p;\vec{x}',t_p')$ on which the derivative operators in $\delta\mathcal{H}^{ab}$ act. A spatial and time diagonal Green function $g_{ss'}(\vec{x},t_p)$ for $G^a_{ss'}$ can be considered in a Thomas-Fermi approximation for large atom masses and a nearly homogeneous system where the kinetic energy and time derivative can be added as a perturbation

$$
\begin{aligned}
G^{a(=1/2)}_{ss'}(\vec{x},t_p;\vec{x}',t_p') &= \langle\vec{x},s;t_p|(\widetilde{\mathcal{H}}+s_D)^{(T)-1}|\vec{x}',s';t_p'\rangle \\
&= \underbrace{\Big(-\imath\varepsilon_p+v(\vec{x},t_p)+s_D(\vec{x},t_p)\Big)^{-1}_{ss'}}_{g_{ss'}(\vec{x},t_p)}\;\delta_{\vec{x},\vec{x}'}\;\delta(t_p-t_p')+\text{derivative terms}\,.
\end{aligned} \tag{43}
$$

The first order term of $\delta\mathcal{H}^{ab}_{ss'}$ in $\mathcal{O}_1$ (second term in 42) vanishes in the Thomas-Fermi approximation with $G^a_{ss'}$ replaced by the spatial and time diagonal function $g_{ss'}(\vec{x},t_p)$ (43). The anomalous terms with the matrix T can be reduced to a 2×2 matrix because the spin space is restricted to the block diagonal densities.

Introducing the following averages of the block diagonal densities with matrix $s_{D;ss'}(\vec{x},t_p)$

$$
\begin{aligned}
\Big\langle\ldots\Big\rangle &= \int \mathrm{d}[s_D]\prod_{\{\vec{x},t_p\}}\frac{w_i^2(\vec{x},t_p)}{4}\Big(\ldots\Big)\;\det\Big(\mathcal{H}+s_D\Big) \\
&\times\;\exp\Big\{-\frac{\imath}{2\hbar}\frac{1}{V_0}\int_C\mathrm{d}t_p\sum_{\vec{x}}\operatorname*{tr}_{s,s'}\Big(s_D(\vec{x},t_p)\Big)^2\Big\}.
\end{aligned} \tag{44}
$$

$$
\frac{\imath}{\hbar V_0}\,c_{tt}(\vec{x},t_p) = \Big\langle\operatorname*{tr}_{s,s'}\Big[(E_p g_{ss'}(\vec{x},t_p))\;(E_p g_{s's}(\vec{x},t_p))\Big]\Big\rangle \tag{45}
$$

$$
\frac{\imath}{\hbar V_0}\,c_{t}(\vec{x},t_p) = \Big\langle\operatorname*{tr}_{s,s'}\Big[(-E_p g_{ss'}(\vec{x},t_p))\;g_{s's}(\vec{x},t_p)\Big]\Big\rangle \tag{46}
$$

$$
\frac{\imath}{\hbar V_0}\,c_{\mu t}(\vec{x},t_p) = \Big\langle\operatorname*{tr}_{s,s'}\Big[(\partial_\mu g_{ss'}(\vec{x},t_p))\;(-E_p g_{s's}(\vec{x},t_p))\Big]\Big\rangle \tag{47}
$$

$$
\frac{\imath}{\hbar V_0}\,c(\vec{x},t_p) = \Big\langle\operatorname*{tr}_{s,s'}\Big[g_{ss'}(\vec{x},t_p)\;g_{s's}(\vec{x},t_p)\Big]\Big\rangle \tag{48}
$$

$$
\frac{\imath}{\hbar V_0}\,c_{\mu}(\vec{x},t_p) = \Big\langle\operatorname*{tr}_{s,s'}\Big[g_{ss'}(\vec{x},t_p)\;(\partial_\mu g_{s's}(\vec{x},t_p))\Big]\Big\rangle \tag{49}
$$

$$
\frac{\imath}{\hbar V_0}\,c_{\mu\nu}(\vec{x},t_p) = \Big\langle\operatorname*{tr}_{s,s'}\Big[(\partial_\mu g_{ss'}(\vec{x},t_p))\;(\partial_\nu g_{s's}(\vec{x},t_p))\Big]\Big\rangle\;, \tag{50}
$$

the coherent state path integral can be separated into an action $\mathcal{S}_0[s_D]$ following from (44) and an action $\mathcal{S}[T,T^{-1};\{c_{ij}\}]$ for the anomalous terms

$$
\begin{aligned}
Z[\mathcal{J}] &\approx \int\mathrm{d}[s_D]\Big(\prod_{\{\vec{x},t_p\}}\frac{w_i^2(\vec{x},t_p)}{4}\Big)\;\exp\Big\{-\imath\,\mathcal{S}_0[s_D]\Big\} \\
&\times\int\mathrm{d}[\phi]\;\exp\Big\{-\imath\,\mathcal{S}[T,T^{-1};\{c_{ij}\}]\Big\}\;\exp\Big\{-\imath\,\mathcal{S}_{\mathcal{J},j}[T,s_D;\mathcal{J},j]\Big\}.
\end{aligned} \tag{51}
$$

The action $\mathcal{S}[T,T^{-1};\{c_{ij}\}]$ is given by the following relation up to second order in the gradients $\hat{\partial}_\mu$, $\hat{E}_p$ where partial spatial integrations have been performed. The parameter functions $c_{ij}(\vec{x},t_p)$

(45-50) contain the properties of the densities with the matrix s_D as a background field

$$\begin{aligned} T^{-1} &= \kappa\, T\, \kappa \qquad\qquad \kappa = \mathrm{diag}(1,-1) && (52)\\ (\mathcal{D}T^{-1}) &= -(E_p T^{-1}) = \imath\hbar\; T^{-1}\;(\partial_{t_p}T)\;T^{-1} && (53)\end{aligned}$$

$$\begin{aligned} S[T,T^{-1};\{c_{ij}\}] = \frac{1}{\hbar V_0}\int_C \mathrm{d}t_p \sum_{\vec{x}} & \qquad (54)\\ & \Big\{ c(\vec{x},t_p)\ \mathop{\mathrm{tr}}_{a,b}\Big[(T\kappa(\mathcal{D}T^{-1}))^2\Big] - c_{\mu\nu}(\vec{x},t_p)\left(\frac{\hbar^2}{m}\right)^2 \mathop{\mathrm{tr}}_{a,b}\Big[(\partial_\mu T)\ (\partial_\nu T^{-1})\Big] \\ + \quad & c_{tt}(\vec{x},t_p)\ \mathop{\mathrm{tr}}_{a,b}\Big[(T\kappa T^{-1}-\kappa)^2\Big] + c_\mu(\vec{x},t_p)\ \frac{2\hbar^2}{m}\ \mathop{\mathrm{tr}}_{a,b}\Big[(\partial_\mu T)\kappa(\mathcal{D}T^{-1})\Big] \\ + \quad & \Big(2\ c_{\mu t}(\vec{x},t_p) - \partial_\mu c_t(\vec{x},t_p)\Big)\ \frac{\hbar^2}{m}\ \mathop{\mathrm{tr}}_{a,b}\Big[[T,\kappa]\ (\partial_\mu T)\Big] \\ + \quad & 2\ c_t(\vec{x},t_p)\ \mathop{\mathrm{tr}}_{a,b}\Big[(T^{-1}-T)\ (\mathcal{D}T)\Big]\Big\}\ . \end{aligned}$$

It can be verified with an expansion of the Goldstone field $\phi(\vec{x},t_p)$ of the matrix T in sinh-amplitude and phase term $\phi(\vec{x},t_p) = \sinh(\varphi(\vec{x},t_p))\ \exp\{\imath\ \alpha(\vec{x},t_p)\}$ ($\varphi(\vec{x},t_p) \geq 0$) that the derived action (54) is composed of a massless Goldstone field $\alpha(\vec{x},t_p)$ and a real massive field $\sinh(\varphi(\vec{x},t_p))$. The parameter functions $c_{ij}(\vec{x},t_p)$ can be regarded as generalized mass and kinetic terms of a spontaneously broken ϕ^4 field theory[24]. They can be calculated from the classical equations (35-38) for $s_D^0(\vec{x},t_p)$ where the Thomas-Fermi approximation of large atom mass can be used for simplicity. In terms of the sinh-amplitude and phase term, the action $S[T,T^{-1};\{c_{ij}\}]$ (54) is similar to that of generalized Sine-Gordon equations

$$\begin{aligned} S[\varphi,\alpha;\{c_{ij}\}] = \frac{1}{\hbar V_0}\int_C \mathrm{d}t_p \sum_x & \qquad (55)\\ & \Big\{2\hbar^2\ \sinh^2(\varphi)\ \Big[c(\vec{x},t_p)\ \cosh(2\varphi)\ (\partial_{t_p}\alpha)^2 + c_{\mu\nu}(\vec{x},t_p)\Big(\frac{\hbar}{m}\Big)^2\ (\partial_\mu\alpha)\ (\partial_\nu\alpha)\Big] \\ + \quad & 2\hbar^2\Big[c(\vec{x},t_p)\ (\partial_{t_p}\varphi)^2 + c_{\mu\nu}(\vec{x},t_p)\Big(\frac{\hbar}{m}\Big)^2\ (\partial_\mu\varphi)\ (\partial_\nu\varphi)\Big] - 8\ c_{tt}(\vec{x},t_p)\ \sinh^2(\varphi) \\ + \quad & \frac{2\hbar^3}{m}\ c_\mu(\vec{x},t_p)\ \sinh(2\varphi)\ \Big[(\partial_{t_p}\alpha)\ (\partial_\mu\varphi) - (\partial_\mu\alpha)\ (\partial_{t_p}\varphi)\Big] \\ - \quad & \frac{4\imath\hbar^2}{m}\Big(2\ c_{\mu t}(\vec{x},t_p) - \partial_\mu c_t(\vec{x},t_p)\Big)\ \sinh^2(\varphi)\ (\partial_\mu\alpha) + 4\imath\hbar\ c_t(\vec{x},t_p)\ \sinh(2\varphi)\ (\partial_{t_p}\varphi)\Big\}\ . \end{aligned}$$

A first order variation of the action (55) with respect to the fields on the time contour gives a classical equation for the time development of the Goldstone field $\overline{\phi}(\vec{x},t) = \sinh(\overline{\varphi}(\vec{x},t))\times\ \exp\{\imath\ \overline{\alpha}(\vec{x},t)\}$ with the densities $s_{D;ss'}(\vec{x},t_p)$ as background fields in the parameter functions $\overline{c}_{ij}(\vec{x},t)$

$$\begin{aligned} \overline{\varphi}(\vec{x},t) &= \frac{1}{2}\Big(\varphi(\vec{x},t_+) + \varphi(\vec{x},t_-)\Big) && (56)\\ \overline{\alpha}(\vec{x},t) &= \frac{1}{2}\Big(\alpha(\vec{x},t_+) + \alpha(\vec{x},t_-)\Big) && (57)\\ \overline{c}_{ij}(\vec{x},t) &= \frac{1}{2}\Big(c_{ij}(\vec{x},t_+) + c_{ij}(\vec{x},t_-)\Big) && (58)\end{aligned}$$

$$\partial_t\Big(\overline{c}\ \sinh^2(\overline{\varphi})\ \cosh(2\overline{\varphi})\ (\partial_t\overline{\alpha})\Big) + \left(\frac{\hbar}{m}\right)^2 \partial_\mu\Big(\overline{c}_{\mu\nu}\ \sinh^2(\overline{\varphi})\ (\partial_\nu\overline{\alpha})\Big) + \qquad (59)$$

$$+ \quad \frac{\hbar}{2m} \sinh(2\overline{\varphi})\Big((\partial_t\overline{c}_\mu)\,(\partial_\mu\overline{\varphi}) - (\partial_\mu\overline{c}_\mu)\,(\partial_t\overline{\varphi})\Big) - \frac{\imath}{m}\partial_\mu\Big((2\overline{c}_{\mu t} - (\partial_\mu\overline{c}_t))\,\sinh^2(\overline{\varphi})\Big) = 0$$

$$\partial_t\Big(\overline{c}\,(\partial_t\overline{\varphi})\Big) + \left(\frac{\hbar}{m}\right)^2 \partial_\mu\Big(\overline{c}_{\mu\nu}\,(\partial_\nu\overline{\varphi})\Big) = \tag{60}$$

$$\sinh(2\overline{\varphi}) \times \Big\{\overline{c}\Big(\cosh(2\overline{\varphi}) - \frac{1}{2}\Big)(\partial_t\overline{\alpha})^2 + \left(\frac{\hbar}{m}\right)^2 \frac{\overline{c}_{\mu\nu}}{2}(\partial_\mu\overline{\alpha})(\partial_\nu\overline{\alpha}) - \frac{2}{\hbar^2}\overline{c}_{tt} - \frac{\imath}{\hbar}(\partial_t\overline{c}_t) +$$

$$- \quad \frac{\hbar}{2m}(\partial_\mu\overline{c}_\mu)(\partial_t\overline{\alpha}) + \Big(\frac{\hbar}{2m}(\partial_t\overline{c}_\mu) - \frac{\imath}{m}(2\overline{c}_{\mu t} - \partial_\mu\overline{c}_t)\Big)\,(\partial_\mu\overline{\alpha})\Big\}_.$$

The equations (59,60) determine the time development of the Goldstone field for the condensate composed of fermionic atoms and are analogous to the Gross-Pitaevskii equation for bosonic constituents.

4 Summary and discussion

Since the parameters $c_{ij}(\vec{x}, t_p)$, following from the densities in the background, change slowly in time and spatial coordinates, the pair condensate is determined by the lowest order terms of the nonlinear sigma model with the matrix T

$$T = \begin{pmatrix} \sqrt{1+|\phi|^2} & \phi^* \\ \phi & \sqrt{1+|\phi|^2} \end{pmatrix}_. \tag{61}$$

This separation into background properties with the parameters c_{ij} and anomalous terms of the nonlinear sigma model corresponds to the observation that only a small fraction of the atoms condense in the Fermi sea. In the case of computations one therefore can introduce correlation functions of the densities, as e.g. the current-current correlation function $c_{\mu\nu}(\vec{x}, t_p)$, etc. which can be calculated with the self-energy matrix $s_D^0(\vec{x}, t_p)$ (35-38) of the first order variation of the action in (34). Taking derivatives of the action $\mathcal{S}_{\mathcal{J},j}[T, s_D; \mathcal{J}, j]$ with respect to $\mathcal{J}$, the appropriate observables can be obtained, as e.g. the anomalous term $\frac{-\imath}{\hbar}\langle\chi_{\vec{x},\uparrow}(t_p)\,\chi_{\vec{x}',\downarrow}(t'_p)\rangle$ of two anticommuting fields is represented by a relation with the bosonic matrix T and the Green function $G^a_{ss'}$ of the densities

$$-\frac{\imath}{\hbar\,\mathcal{N}}\,\langle\chi_{\vec{x},\uparrow}(t_p)\,\chi_{\vec{x}',\downarrow}(t'_p)\rangle = \tag{62}$$

$$= \sum_{a=1}^{2}(T^{-1})^{2a}(\vec{x}', t'_p)\,\Big[G^a_{\downarrow\uparrow}(\vec{x}', t'_p; \vec{x}, t_p) - G^a_{\uparrow\downarrow}(\vec{x}, t_p; \vec{x}', t'_p)\Big]\,T^{a1}(\vec{x}, t_p) \quad .$$

Statements about binding energies of the pairs $\langle\chi_{\vec{x},\uparrow}(t_p)\,\chi_{\vec{x},\downarrow}(t_p)\rangle$ can be calculated by various saddle point considerations from the coherent state path integral (24) [25, 26]. However, the exact expression of the saddle point is not needed for the derivation of the nonlinear sigma model with actions (54) and (55) for the time development of a condensate composed of fermionic atoms. The equations (59) and (60) simplify considerably for a translation invariant system with a spatially constant and time independent external potential $u(\vec{x}, t)$ (5) so that only terms with the constant coefficients $\overline{c}$ (48) and $\overline{c}_{\mu\nu}$ (50) remain. In this case a simpler form of the Sine-Gordon equation results for the condensate of the Goldstone field ϕ in the matrix T (61).

In experiments fermionic ^{6}Li was cooled to degeneracy by ^{23}Na or by mixing two different states in an optical trap [27, 28] and fermionic ^{40}K was cooled to $T/T_F = 0.3$, $(E_F = k_B T_F)$ by sympathetic cooling with ^{87}Rb [29]. Apart from degeneracy a BCS transition, for which the time development of the condensate is described by Eqs. (44) to (60), is suggested in Ref. [30]. For this a degenerate Fermi gas with attractive interaction between the fermions must be prepared. A

possible realization can be obtained in fermionic ^{6}Li [30]. In experiments atoms can be prepared in two different hyperfine states and a Feshbach resonance is used to tune the interaction due to elastic collisions to the desired value [31]. In Refs. [32]-[34], BCS transition temperatures T_{BCS} are predicted in the range from $T_{BCS}/T_F = 0.025$ to $T_{BCS}/T_F = 0.4$, but it is an open question if these conditions can be reached experimentally.

Close to a Feshbach resonance, the additional phenomenon of a BCS to BEC crossover is predicted for atomic Fermi gases [35]. A strong attractive interaction can arise between fermions mediated by bosonic quasi-molecules associated with a Feshbach resonance. In this case it is stated that one has to be careful in applying pure BCS theory to a Fermi gas when the pairing interaction is very strong [35], due to fluctuations in the two-particle Cooper channel. One can also try to extend the experiments in optical lattices with bosonic constituents to fermionic atoms [36]. However, optical lattices involve the additional length scale of the periodic potential so that one has to examine whether a gradient expansion up to second order is sufficient for describing the atoms interacting on one lattice site. In this case one has to take into account the band structure of the optical lattice in the expansion of the Tr ln-term of relation (39).

In the present paper we have performed symmetry considerations on which the derivation of the nonlinear sigma model with the Goldstone field ϕ in the coset matrix T (61) is based. Since the Sine-Gordon equation allows nontrivial solitonic solutions in 1+1 or two spatial dimensions, one can also expect solitons in condensates composed of fermions and study their temporal evolution. Apart from numerical computations, the derived effective Gross-Pitaevskii equation for fermions gives rise to investigations for Bäcklund-transformations in 1+1 or two spatial dimensions so that nontrivial solutions can be obtained as for the nonlinear Schrödinger equation of bosonic condensates.

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A Symplectic symmetry of the generating function

In the following we abbreviate spatial and time coordinates $\vec{x}$, t_p of the fields $\chi_{\vec{x},s}(t_p)$, $\chi^*_{\vec{x},s}(t_p)$ with the indices i, j in the global transformations acting in the doubled spin space. The symmetries result from the spin independent one particle Hamiltonian $\widetilde{\mathcal{H}}$ and the spin independent interaction. Considering the Hamiltonian $\widetilde{\mathcal{H}}_{ij}$ as a spin independent matrix, we can rewrite the bilinear Hamiltonian form with the antisymmetry of the fields $\chi_{i,s}$, $\chi^*_{i,s}$ as the relation

$$\begin{aligned}
\chi^*_{i,s}\;\widetilde{\mathcal{H}}_{ij}\;\chi_{j,s} &= \frac{1}{2}\Big(\chi^*_{i,s}\;\widetilde{\mathcal{H}}_{ij}\;\chi_{j,s}-\chi_{j,s}\;\widetilde{\mathcal{H}}_{ij}\;\chi^*_{i,s}\Big) \\
&= \frac{1}{2}\begin{pmatrix}\chi^*_{i,s}\\ \chi_{i,s}\end{pmatrix}^T\begin{pmatrix}\widetilde{\mathcal{H}}_{ij} & 0\\ 0 & -\widetilde{\mathcal{H}}^T_{ij}\end{pmatrix}\begin{pmatrix}\chi_{j,s}\\ \chi^*_{j,s}\end{pmatrix}.
\end{aligned} \tag{63}$$

After a transformation with a matrix M, one has to obtain again a separation into fields $\chi'_{i,s}$ and their complex conjugates $\chi^{*\prime}_{i,s}$ in the first, second and third, fourth row. Therefore, the global transformations with the 4×4 matrix M consist of two 2×2 block matrices A, B in spin space and their complex conjugates

$$\underbrace{\begin{pmatrix}\chi'_{i,s'}\\ \chi'^*_{i,s'}\end{pmatrix}}_{\eta'_i}=\underbrace{\begin{pmatrix}A_{s's} & B_{s's}\\ B^*_{s's} & A^*_{s's}\end{pmatrix}}_{M}\underbrace{\begin{pmatrix}\chi_{i,s}\\ \chi^*_{i,s}\end{pmatrix}}_{\eta_i}\quad. \tag{64}$$

Because of relation (63) the matrix M has to be invariant under the following unitary hyperbolic transformation

$$M^+\;\kappa\;M=\kappa \qquad \kappa=\begin{pmatrix}1_2 & \\ & -1_2\end{pmatrix} \tag{65}$$

or the equivalent symplectic form which results from interchanging in κ the first and second row with the third and fourth row, respectively, and from reordering the matrix M^+ to its transpose M^T

$$M^T\;g\;M=g \qquad g=\begin{pmatrix}0 & -1_2\\ 1_2 & 0\end{pmatrix}. \tag{66}$$

The bilinear form $\chi^*_{i,\uparrow}\chi_{i,\uparrow}+\chi^*_{i,\downarrow}\chi_{i,\downarrow}$ is invariant under these transformations with the matrix M, both for the unitary hyperbolic and symplectic kind of transformation

$$\begin{aligned}
2\,(\chi^*_{i,\uparrow}\chi_{i,\uparrow}+\chi^*_{i,\downarrow}\chi_{i,\downarrow}) &= \eta^+_i\;\kappa\;\eta_i=\eta^T_i\;g\;\eta_i \\
&= \eta'^+_i\;\kappa\;\eta'_i=\eta'^T_i\;g\;\eta'_i \\
&= 2\,(\chi'^*_{i,\uparrow}\chi'_{i,\uparrow}+\chi'^*_{i,\downarrow}\chi'_{i,\downarrow})\;\;.
\end{aligned} \tag{67}$$

The complex 2×2 matrices A, B consist of sixteen real parameters which are restricted by the equations (65) or (66), yielding six real conditional relations. Therefore, ten independent parameters remain in the matrices A, B. This indicates a symplectic symmetry group $Sp(N/2)$ with $\frac{1}{2}N(N+1)$ parameters for $N=4$ of the 4×4 global transformation matrix M. The matrix M can be expressed with the global $U(2)$ subgroup in spin space and the coset space $Sp(2)\backslash U(2)$ which is composed of the complex symmetric matrices $t_{ss'}$ or $t'_{ss'}$ with six and ten real parameters, respectively

$$\begin{aligned}
M &= \begin{pmatrix}U^+ & \\ & U^T\end{pmatrix}\begin{pmatrix}\sqrt{1+t^+\,t} & t^+\\ t & \sqrt{1+t\,t^+}\end{pmatrix}\begin{pmatrix}U & \\ & U^*\end{pmatrix} \\
&= \begin{pmatrix}\sqrt{1+t'^+\,t'} & t'^+\\ t' & \sqrt{1+t'\,t'^+}\end{pmatrix}
\end{aligned} \tag{68}$$

$$t' = U^T\,t\,U \qquad t=t^T \qquad t'=t'^T\;. \tag{69}$$

The matrices A, B and their complex conjugates are therefore given by the following equations which contain ten real independent parameters

$$A = \sqrt{1 + t'^{+}\, t'} \qquad B = t'^{+} \tag{70}$$

$$A^{*} = \sqrt{1 + t'\, t'^{+}} \qquad B^{*} = t' \ . \tag{71}$$

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